

Convergence rate for Ehrenfest-type urn designs ^(*)

Velocità di convergenza per esperimenti markoviani: le urne degli Ehrenfest

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Riassunto: In questo lavoro si analizza il problema della velocità di convergenza per esperimenti markoviani generati attraverso processi di tipo Ehrenfest, ossia opportune estensioni del classico modello ad urne introdotto dai coniugi Ehrenfest nell'ambito della meccanica statistica. Si definiscono markoviani quegli esperimenti sequenziali in cui il ricercatore decide ad ogni passo come effettuare l'osservazione successiva sulla base della sola osservazione più recente. Tali procedure possono essere adeguatamente studiate per mezzo di una catena di Markov, la cui distribuzione stazionaria descrive completamente il comportamento asintotico dell'esperimento. Sulla base di risultati per passeggiate aleatorie ergodiche è possibile fornire alcune condizioni che permettono di caratterizzare in modo esatto la velocità di convergenza della catena alla distribuzione stazionaria.

Keywords: Markov chains, sequential designs, convergence rates, Ehrenfest processes.

1. Introduction

Assume we want to carry out a sequential experiment for comparing the efficacy of two treatments T_1 and T_2 . The peculiar feature of sequential designs is that they employ the data collected in the past of the experiment in order to choose next allocations, called *design points*. These procedures are essential in clinical trials and more in general in all the situations for which the entrance of statistical units is naturally sequential.

Usually, the experimenter may have several objectives in mind (inference about population parameters, cost saving, ethical purposes ...) and often this leads to a given proportion π_1^* of allocations to T_1 which can be regarded as optimal. For instance, assuming a linear homoscedastic model for the responses under T_1 and T_2 , if we are interested in estimating the difference between the mean responses with maximum precision then the balanced allocation minimizes the variance of the estimator. Moreover, some form of randomization is required to protect the experiment against various types of bias and urn models provide a flexible tool for achieving a random assignment of the treatments.

In the following we assume that the target proportion of allocations to the two treatments is known in advance; in particular, we will focus on a generalization of the Ehrenfest design, proposed by Chen (2000), called the Ehrenfest-type urn design, recently introduced in Costantini and Garibaldi (2003) and in Baldi Antognini (2004). It is a class of Markovian procedures (as defined in Giovagnoli (2004) generated by suitable extensions of the Ehrenfest urn process designed to achieve almost sure convergence to any given target. The aim of this paper is to analyze the convergence of Ehrenfest-type urn

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designs in order to characterize the speed of convergence towards its target. The results obtained allow us to determine suitable conditions on the urn process for improving the convergence rate. Furthermore, for any fixed target, and within the class of the Ehrenfest-type urn designs, we show that it is always possible to find the fastest procedure.

2. Ehrenfest-type urn design for the allocation of two treatments

It is a class of sequential procedures for comparing two treatments, recently introduced by Baldi Antognini (2004) which can be briefly described as follows. Assume there are 2 urns U_1, U_2 containing an overall given number w of balls; statistical units enter sequentially the trials and for each of them one ball is drawn at random from the w balls of the urns. If the ball that has been drawn belongs to urn U_i , then treatment T_i is allocated to the next subject ($i = 1, 2$). At each stage n , let $W(n) = (W_1(n), W_2(n))$ be the urn composition and $\delta_1(n)$ denotes the allocation of T_1 , that is, $\delta_1(n) = 1$ if the n -th unit is allocated to T_1 , 0 otherwise. Also, let $N_1(n)$ be the number of allocations to the first treatment, where $N_1(n) = \sum_{k=1}^n \delta_1(k)$. At each stage n only one ball is drawn and replaced, then the initial number w of balls is kept constant, namely $\sum_{i=1}^2 W_i(n) = w$, so that the probability of allocating T_1 at the step $(n + 1)$ is given by:

$$\Pr(\delta_1(n + 1) = 1 \mid W(n)) = \frac{W_1(n)}{w}. \quad (2.1)$$

Furthermore, the ball that has been drawn is returned to one of the two urns and the replacement rule depends only both on the current urn composition and on the latest treatment allocation:

$$\begin{aligned} \Pr(W_1(n + 1) = i + 1 \mid \delta_1(n + 1) = 0; W_1(n) = i) &= a(i) \\ \Pr(W_1(n + 1) = i - 1 \mid \delta_1(n + 1) = 1; W_1(n) = i) &= b(i) \end{aligned} \quad (2.2)$$

where $i \in \mathcal{X} = \{0, \dots, w\}$ and $a(w) = b(0) = 0$. Clearly, each process $\{W_1(n)\}$ or, equivalently, any Ehrenfest-type urn design, can be identified by the specification of the functions $a(\cdot)$ and $b(\cdot)$, where $a(\cdot), b(\cdot) : \mathcal{X} \rightarrow [0, 1]$. Notice that, as shown in Baldi Antognini and Giannerini (2006), for any choice of $a(\cdot)$ and $b(\cdot)$, $\{W_1(n)\}$ is an ergodic Markov chain on $\mathcal{X}$ with transition probabilities

$$\Pr(W_1(n + 1) = j \mid W_1(n) = i) = \begin{cases} p_i = a(i) \left(1 - \frac{i}{w}\right) & j = i + 1 \\ r_i = 1 - p_i - q_i & j = i \\ q_i = b(i) \left(\frac{i}{w}\right) & j = i - 1. \end{cases} \quad (2.3)$$

The next theorem highlights the relationship between a given Ehrenfest-type urn design and the associated urn process and allow us to find asymptotically optimal procedures, namely designs for which, given a target proportion π_1^* of allocations to T_1 , it results $N_1(n)/n \rightarrow_{n \rightarrow \infty} \pi_1^*$ a.s.. Now, let ξ be the stationary distribution of the Markov chain $\{W_1(n)\}$ and let E_ξ denote its ergodic mean.

Theorem 2.1. *An Ehrenfest-type urn design is asymptotically optimal if and only if the corresponding ergodic chain $\{W_1(n)\}$ satisfies the condition $E_\xi = w \pi_1^*$.*

3. Some results on a class of random walks

The following Theorem show that for a given class of random walks it is possible to derive a recursion on the first moment, so that, in addition to the available asymptotics, these chains can be characterized exactly at each step.

Theorem 3.1. *Let $\{W_1(n)\}$ be a time homogeneous random walk on $\{0, 1, \dots, w\}$ with:*

$$P_{ij} = \begin{cases} p_i & j = i + 1 \\ r_i & j = i \\ q_i & j = i - 1 \end{cases}$$

where $p_i + r_i + q_i = 1$ for $i = 1, \dots, w - 1$ and $q_0 = p_w = 0$. If

$$p_i - q_i = p_0 - \frac{i}{w}(p_0 + q_w) \quad i = 1, \dots, w - 1 \quad (3.1)$$

then

1. $E_\xi = w \left(\frac{p_0}{p_0 + q_w} \right)$
2. $|E[W_1(n)] - E_\xi| = \left| E[W_1(0)] - w \left(\frac{p_0}{p_0 + q_w} \right) \right| \left(1 - \frac{p_0 + q_w}{w} \right)^n$

Consequently, $E[W_1(n)] \rightarrow E_\xi$ geometrically fast with rate $\rho = \left(1 - \frac{p_0 + q_w}{w} \right)$.

Note that, under hypothesis (3.1), both the ergodic mean E_ξ and the rate of convergence ρ depend on the transition matrix only through the boundary probabilities p_0 and q_w .

4. Convergence for Ehrenfest-type urn designs

Under any Ehrenfest-type urn design, from the fundamental Lemma in Giovagnoli (2004) we are able to assess its performance through the analysis of the properties of the associated ergodic Markov chain, since the asymptotic behaviour of such a procedure is completely determined by the asymptotic behaviour of the corresponding urn process.

From now on, we consider only Ehrenfest-type urn procedures for which the corresponding Markov chain $\{W_1(n)\}$ satisfies condition 3.1 and we denote this class of procedures by Υ .

Theorem 4.1. *For any given target π_1^* , an Ehrenfest-type urn design in Υ is asymptotically optimal if and only if*

$$b(w) = a(0) \left(\frac{1 - \pi_1^*}{\pi_1^*} \right) \quad \text{and} \quad w - i = \frac{b(i) - b(w)}{a(i) - a(0)} \quad i = 1, \dots, w - 1. \quad (4.1)$$

4.1. Convergence to the stationary distribution

From Theorem 3.1, under any Ehrenfest-type urn design in Υ the sequence of the expected values of the corresponding urn process converges monotonically to the ergodic mean with geometric order. Recently, the quantity

$$S = \sum_{n=1}^{\infty} |E[W_1(n)] - E_\xi| \quad (4.2)$$

has been proposed in Coolen-Schrijner and Van Doorn (2000) as an indicator of the rate of convergence to the stationary distribution ξ for an ergodic Markov chain $\{W_1(n)\}$, provided that the sequence of the expected values $\{E[W_1(n)]\}$ converges monotonically to its limit E_ξ . Clearly, the convergence is faster as the value of S decreases. From Theorem 3.1 it is possible to derive exactly the speed of convergence towards stationarity.

Corollary 4.2. *Given a target π_1^* , for any asymptotically optimal Ehrenfest-type urn design in Υ it results*

$$S = \left| W_1(0) - w \frac{a(0)}{a(0) + b(w)} \right| \left(\frac{w}{a(0) + b(w)} - 1 \right).$$

Thus, the convergence towards stationarity is faster as the value of $a(0)$ and $b(w)$ increase or the number w of balls decreases. From condition (4.1), for any fixed target π_1^* we have

$$\max_{\Upsilon} \left\{ \frac{a(0) + b(w)}{w} \right\} = \max_{\Upsilon} \left\{ \frac{a(0)}{w\pi_1^*} \right\} \quad (4.3)$$

which imply to let $a(0) = 1$.

Remarkably, it is possible to show that each urn process in Υ has the generalized negative Hypergeometric distribution as stationary law (see Baldi Antognini 2005, Baldi Antognini and Giannerini 2006). In some cases (e.g. Chen's design, where $a(i) = b(i) = 1 \forall i \in \mathcal{X}$) through the Hahn orthogonal polynomials representation (see Seneta 2001) the whole spectrum of eigenvalues of the transition matrix of the corresponding chains is available in closed form. The largest absolute value of the non-trivial eigenvalues, which is known to measure the convergence towards the stationary distribution, corresponds exactly to the rate of convergence of the moments given in Theorem 3.1 (see Baldi Antognini and Giannerini 2006, for details).

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